Problem Solving Strategies

The basic problem solving approach used in many mathematic classrooms is

Step 1: Understand the problem.
Step 2: Consider possible solution strategies and plan an approach.
Step 3: Carry out one strategy.
Step 4: Look back: Does the solution make sense?

However, most students who haven't become proficient at solving word problems get stuck on the very first step, understanding. So a great deal of problem solving effort needs to go into understanding the problem.

Strategy 1: Well-structured problems

Problem solving begins in kindergarten (or earlier) with simple, well-structured problems that connect to a child's everyday understanding of how things work. Typical early problems involve joining or separating quantities, and they are structured so that children can easily model the action in a problem, using counters.

Jennie has 3 shells. Her brother gives her 5 more shells. Now how many shells does Jennie have? (joining 3 shells and 5 shells)

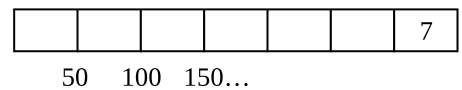
Pete has 6 cookies. He eats 3 of them. How many cookies does Pete have then? (separating 3 cookies from 6 cookies)

8 birds are sitting on a tree. Some more fly up to the tree. Now there are 12 birds in the tree. How many flew up? (joining, where the change is unknown)

The types of well-structured problems get progressively more complicated, and the solutions that children develop to solve them get progressively more sophisticated, leading eventually to fluency and problem solving competence. Other types include part-whole, comparison, rate, array, and grouping problems. (See Carpenter et. al., 1999)

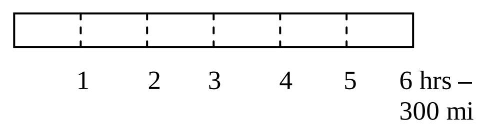
Algebra and geometry problems can also be structured to follow a pattern, making the solution approach more apparent. Here are some examples from algebra:

1. *You're driving on a vacation. You drive at 50 mph for 7 hours. How far have you driven?*



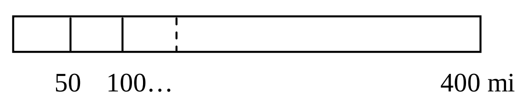
Counting out 7 groups of 50: d = 50 · 7
This procedure generates the formula
$d = r \cdot t$

2. *You drive 300 miles in 6 hours. How fast were you driving, on average? (how many miles do you go in each hour)*



Dividing 300 into 6 groups: r = 300/6
What formula would you create from this example? Can students see how this relates to $d = r \cdot t$ *?*

3. *How long does it take you to drive 400 miles at 50 mph?*



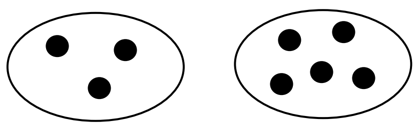
Counting how many 50's in 400 (measurement division). 400 / 50 = ___
How does the formula relate to this calculation?

Whenever you introduce new types of real-life problems, structure them carefully to allow students to build on what they already know.

Strategy 2: Objects – Drawings – Symbols

Young children solve well-structured problems by using objects – counters – to model the action in the problem. For the *Jennie* problem, a child will count out 5 shells, and then bring forward 3 more. At first, many children re-count them all. Over time, they develop more efficient strategies, like counting on from the first number, or counting on from the larger number.

The next step in children's strategies is to abstract the action in the problem into a simple picture.

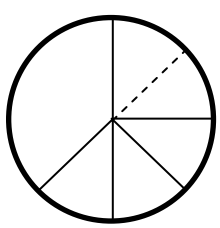


And from here, students learn to write the action in mathematical symbols:

$$3 + 5 = 8$$

This object – drawing – symbols progression works for learning all kinds of mathematics, including fractions, decimals and percents. When older students struggle with setting up a problem symbolically, have them back up and use a drawing first to represent the situation. The drawing gives you good insight into their thinking.

Try it: Use fraction bars or fraction circles to model 1/4 + 3/8.



Then make a drawing to show how to add them.

Using the process involved in the drawing, develop a procedure that can be used with the symbolic expression. In this case, to count the pieces in your drawing, you need to recognize that 1/4 is the same as 2/8. The procedure then is to create common denominators, then add the numerators, which represent the number of pieces.

Strategy 3: Think-Aloud (from Davey, 1983)

You can help students become strategic readers and problem solvers by modeling the thinking *you* use to make sense of a problem. As students listen to the questions you ask yourself and the ways you guide your own reading, they learn to ask themselves questions to guide their own sense-making.

The "think-aloud" process is usually introduced in four steps, gradually transferring responsibility to students:

1. The teacher reads a problem and stops as needed to explain her thoughts. Students listen. They all solve the problem together.
2. The teacher reads the problem and stops often. Students express their thoughts at each point (and often write them). The whole class, led by the teacher, solves the problem together.
3. The teacher reads the problem, allowing students to signal stopping points as thoughts occur to them. Students solve the problem individually, and then discuss their interpretations of it and solution strategies.
4. Students do this together, in pairs. They work together to solve the problem.

Try it: Use these problems with a partner to try the first and second steps in the think-aloud process.

> *The boys swim team and the girls swim team held a car wash. They made $210 altogether. There were twice as many girls as boys, so they decided to give the girls' team twice as much money as the boys' team. How much did each team get?*

A benefit of this strategy for students who need additional support is that they also listen to the thinking of their classmates, enabling them to learn additional strategies from each other.

Strategy 4: K-W-C

Arthur Hyde and colleagues (2006) have adapted the K-W-L strategy to use in mathematics problem solving.

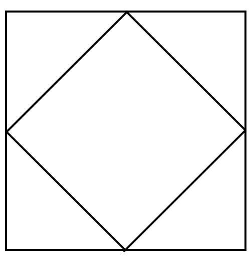
K – What do we know?
W – What do we want to find out?
C – What are the constraints – special conditions, tricks to watch out for?

Try it with a problem:

> *How many different ways can a person make change for a quarter?*

These questions help students think about the problem, to produce an understanding of what's going on and what they have to figure out. They help students read the problem carefully, asking questions of the text, themselves, and the author.

Here's another problem to try, using K-W-C:



Square within a square:

> *The outer square was marked at the midpoints of each side. Then the midpoints were connected to form an inner square.*
>
> *What's the relationship between the area of the inner square and the area of the outer square?*

- What do you know?
- What do you want to find out?
- What are the special conditions or constraints in this problem?

Strategy 5: The "No Numbers" Strategy

Sometimes students can conceptualize a problem more easily when the numbers are removed. Try it with this problem:

> *Leroy paid a total of $23.95 for a pair of pants. That included the sales tax of 6%. What was the price of the pants before the sales tax?*

A reading of this problem without the numbers might be something like this:

> *Leroy bought some pants. The sales tax was included in total price. How much did the pants cost before the tax?*

Then a student might say "I see that we need to subtract!"

A note on traditional problem solving: Why the "key word" strategy isn't always helpful.

To help students through the sticky part of understanding a problem, teachers or textbooks sometimes try to provide a shortcut, often the "key word" strategy. How well does it work in the following problem?

> *The sum of a number and 25 is 43. What's the number?*

We often teach students to solve these kinds of problems using the "translation" method. While this will work better for this problem than a "key word" approach, it still doesn't help students think through the problem, so they don't learn how to transfer what they learn in this problem to new problems.

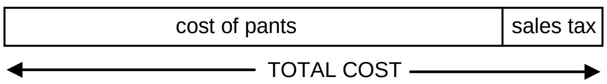
We really want students to learn to think about situations. Thinking produces understanding. Think about a similar problem to the one above:

> *Tina has 25 crayons. Jim gives her some more. Now she has 43 crayons. How many crayons did she get from Jim?*

Figure out two or three different ways to solve this. Did any of them involve a key word?

Strategy 6: The Bar Model

This "no numbers" approach (Strategy 5) lends itself to graphic representations too, like the one below.

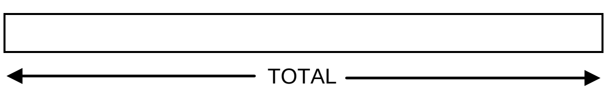


Once the problem is understood in terms of the bar model, students can put the numbers back in and solve it – understand, plan, execute, look back.

What would a bar model look like for the swim team problem?

> *The boys swim team and the girls swim team held a car wash. They made $210 altogether. There were twice as many girls as boys, so they decided to give the girls' team twice as much money as the boys' team. How much did each team get?*

Always start with the total amount represented by the whole bar:



Strategy 7: Number – Unit – Label

Teachers often report that many students simply refuse to attempt assessment items when there is a significant amount of reading, such as this item from an ACT practice test:

> *A hot-air balloon 70 meters above the ground is falling at a constant rate of 6 meters per second while another hot-air balloon 10 meters above the ground is rising at a constant rate of 15 meters per second. To the nearest tenth of a second, after how many seconds will the 2 balloons be the same height above the ground?*

A simple strategy students can use for longer word problems such as these is to list each of the numbers in the problem, the unit for that quantity, and a label for that quantity.

number	**unit**	**label**
70	meters	height of balloon 1
6	meters per second	rate of falling of balloon 1
10	meters	height of balloon 2
15	meters per second	rate of rising of balloon 2
1/10	second	accuracy of answer
2	balloons	I knew that already

Listing the numbers, units and labels helps students lay out the problem in pieces. In this problem, the number "1/10" gives them a clue about what they should find in the problem (see the K-W-C strategy to identify "What you want to find out.")

The next step they choose depends on their problem-solving preference. They might make a drawing (see Strategy 9); they might make a table of values and solve it that way; they might draw a graph; they might make a single equation or a set of simultaneous equations.

To get students started on this strategy, you'll have to talk it through with them several times (see the Think-Aloud strategy). Eventually you want them to use this kind of chart on their own to help them understand and set up the problem.

Strategy 8: Differentiate Instruction

When groups of students are at different places in their learning, you can customize problems based on their readiness for moving ahead:

To do this, start by identifying the learning target for the assignment. Then write a typical "grade level" task that is interesting, challenging and requires students to use the learning target. After studying the pre-assessment data to identify the needs of struggling and advanced learners, adjust at least one aspect of the assignment as needed. This can be:

- the process (simple to complex)
- the content (difficult numbers or simple ones)
- the resources needed to solve the problem
- or the products (how the solution will be presented).

Here's an example from *When Students Choose the Challenge*, by David Suarez in Educational Leadership, Nov. 2007:

Lesson Topic: Problem solving with linear equations

Green-level task (foundational)
The difference in the ages of two people is 8 years. The older person is 3 times the age of the younger. How old is each?

Blue-level task (intermediate)
The length of a rectangle is 3 less than half the width. If the perimeter is 18, find the length and width.

Black-level task (advanced)
When asked for the time, a problem-posing professor said, "if from the present time, you subtract one-sixth of the time from now until noon tomorrow, you get exactly one-third of the time from noon until now." What time was it?

You can assign these problems to students to meet their learning levels, or you can let them "choose the challenge." David Suarez reports that when students are allowed to choose their level of challenge, they most often choose a harder task than you would have assigned to them.

Strategy 9: Re-write the problem

Here's a writing strategy that can help students comprehend better what they are reading. Ask students to read a story problem, and then re-write it from a different point of view, a different "voice." They could write it from the point of view of one of the numbers. Or they could put themselves into the problem. Not only would they have fun with it, but it would give you great insight into how they are reading and interpreting the problem. Have them share their new problems with each other.

Try it:

> *A small plane carrying three people makes a forced landing in the desert. The people decide to split up and go in three different directions in search of an oasis. They agree to divide equally the food and water they have, which includes 15 identical canteens, 5 full of water, 5 half-full of water, and 5 empty. They will want to take the empty canteens with them in case they find an oasis. How can they equally divide the water and the canteens among themselves?* (Hyde, p. 31)

Strategy 10: Make a drawing

Some students are much more visually-oriented, so drawing the situation or action in a problem might come more naturally for them than re-writing. The drawing doesn't have to be a schematic diagram – it can be an illustration of the action in the problem, or a cartoon with panes depicting the situation.

When you let students re-write or re-draw the problem, you are allowing them to bring all their creative, right-brain strategies into the effort to understand the problem!

Example: Daniels and Zemelman, p. 121

The Five Most Powerful Representational Strategies (Hyde, 2006)

"The two common strategies of looking for a pattern and using logical reasoning *always* should be used in problem solving… Mathematics is the science of patterns… Logical reasoning is essential to doing mathematics." (p. 10)

Along with looking for patterns and using logical reasoning, there are five very powerful representational strategies.

- Discuss the problem in small groups (language representations using auditory sense).
- Use manipulatives (concrete, physical representations using tactile sense).
- Act it out (representations of sequential actions using bodily kinesthetic sense).
- Draw a picture, diagram or graph (pictorial representations using visual sense).
- Make a list or table (symbolic representations often requiring abstract reasoning).

Try it: Create a mathematical representation of the situation below. Choose the type of representation you want to use.

> *Amanda's grandmother said that she will put money into a savings account for Amanda if Amanda will also save some money. She said she would put twice as much money as Amanda saves, plus $12, each time Amanda adds money to her savings account.*

Here's another problem to try:

> *At the beginning of the day, 3 people know a secret. Each person tells a new person the secret every hour. Therefore, the number of people who know the secret doubles every hour.*

How many representations can you make for this problem? Would it be easier for some of your students to start with one kind of representation rather than another?

The Value of Teaching with Problems

- Problem solving places the focus of the students' attention on ideas and sense making.
- Problem solving develops the belief in students that they are capable of doing mathematics and that mathematics makes sense.
- Problem solving provides ongoing assessment data that can be used to make instructional decisions, help students succeed, and inform parents.
- Problem solving allows an entry point for a wide range of students. Good problem-based tasks have multiple paths to the solution.
- A problem-based approach engages students so that there are fewer discipline problems.
- Problem solving develops "mathematical power."
- It is a lot of fun!

From J. Van de Walle, 2007, *Elementary and Middle School Mathematics: Teaching Developmentally*, Pearson/Allyn & Bacon, p. 39

Do students care? Use problems set in contexts that are interesting to them.

Proponents of differentiated instruction say that we can create powerful environments for learning when we use tasks that are interesting to students, that allow them to use their preferred learning style, and that are challenging but not too hard. Students will obviously be more willing to engage with a math problem if it is connected to their own life, or if it is inherently interesting for some other reason.

Find out what your students are interested in, and change the context of problems to fit their interests. If it's sports, or music, or clothes, or animals… whatever. Or let them create problems from their own areas of interest.

Try to put 12½ ÷ ¾ into different contexts.

Projects

Larger problems are often more interesting to students because they are more significant.

One typical project uses a scenario of a school that is building a playground. Families in the school will gather on a weekend to put it together. The students need to plan for the supplies that are needed. A list of parts is given, how many of each part, and the cost of each part – or variations on that kind of information. Students then calculate how much the playground will cost, perhaps adjusting the size of the playground to their budget.

Projects like this can be as complex or straightforward as fits your students.

What projects do you use?

Building Confidence through Success

The key to keeping students engaged is to move them gradually through the learning progression, giving them the support they need to move ahead. This allows them to experience success at each step of the way, and success breeds confidence.

References

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Arthur Hyde (2006). *Comprehending Math: Adapting Reading Strategies to Teach Mathematics.* Heinemann, 2006

John A. Van de Walle (2007). *Elementary and Middle School Mathematics: Teaching Developmentally,* Pearson/Allyn & Bacon, p. 39

Questions you can ask to help figure out math problems or textbooks:

How does this connect to my everyday life?

What do I know about this already that might be helpful?

Can I simplify this in some way to see a pattern?

What are the big ideas in this passage or problem?

Can I visualize what's going on here? Can I draw a picture of this situation that might help me see the connections among quantities?

What are the assumptions or the givens?

What am I trying to find out (a problem), or what are they trying to teach me (text)?

What words don't I know – who can I ask?

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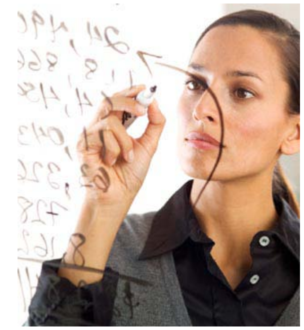

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